Limit of functions (solution)

1. **(a)** Put $y = \pi - x$. As $x \to \pi$, $y \to 0$.

$$\lim_{x\to\pi}\frac{\sin mx}{\sin nx}=\lim_{y\to 0}\frac{\sin m(\pi-y)}{\sin n(\pi-y)}=\frac{(-1)^{m-1}}{(-1)^{n-1}}\lim_{y\to 0}\frac{\sin my}{\sin ny}=\frac{m}{n}(-1)^{m-n}\lim_{y\to 0}\frac{\sin my/my}{\sin ny/ny}=\frac{m}{n}(-1)^{m-n},\ m,n\in\mathbf{N}$$

(b) $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}$$

(c) Put $y = 1 - x$. As $x \to 1$, $y \to 0$.

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\lim_{y\to 0}y\tan\frac{\pi(1-y)}{2}=\lim_{y\to 0}y\cot\frac{\pi y}{2}=\frac{2}{\pi}\lim_{y\to 0}\frac{\frac{\pi y}{2}}{\tan\frac{\pi y}{2}}=\frac{2}{\pi}$$

(d) $$\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=\lim_{x\to 0}\frac{2in2x\sin x}{x^2}=4\lim_{x\to 0}\frac{\sin 2x}{2x}\lim_{x\to 0}\frac{\sin x}{x}=4$$

(e) $$\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}=\lim_{x\to\frac{\pi}{6}}\frac{(2\sin x-1)(\sin x+1)}{(2\sin x-1)(\sin x-1)}=\lim_{x\to\frac{\pi}{6}}\frac{\sin x+1}{\sin x-1}=\frac{(1/2)+1}{(1/2)-1}=-3$$

(f) Put $y = x - (\pi/3)$. As $x \to (\pi/3)$, $y \to 0$.

$$\lim_{x\to\frac{\pi}{3}}\frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}=\lim_{x\to 0}\frac{\sin y}{1-2\cos\left(y+\frac{\pi}{3}\right)}=\lim_{x\to 0}\frac{\sin y}{1-2\left(\cos y\cos\frac{\pi}{3}-\sin y\sin\frac{\pi}{3}\right)}$$

$$=\lim_{x\to 0}\frac{\sin y}{1-\cos y+\sqrt{3}\sin y}=\lim_{x\to 0}\frac{\sin y}{2\sin^2(y/2)+\sqrt{3}\sin y}=\lim_{x\to 0}\frac{\frac{\sin y}{y}}{\frac{\sin(y/2)}{y/2}\sin(y/2)+\sqrt{3}\frac{\sin y}{y}}=\frac{1}{\sqrt{3}}$$

2. **(a)** $$\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=\frac{0-1}{2\times 0-0-1}=1$$

(b) $$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(2x+1)}=\lim_{x\to 1}\frac{x+1}{2x+1}=\frac{2}{3}$$

(c) $$\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}}=\frac{1-0}{2-0-0}=\frac{1}{2}$$

3. **(a)** $$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=\lim_{x\to 0}\frac{6x^3+11x^2+6x}{x}=\lim_{x\to 0}\left[6x^2+11x+6\right]=6$$

(b) $$\lim_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2+x^5}=\lim_{x\to 0}\frac{10x^2+10x^3+5x^4+x^5}{x^2+x^5}=\lim_{x\to 0}\frac{10+10x+5x^2+x^3}{1+x^3}=10$$

(c) $x + x^2 + \dots + x^n - n = (x-1) + (x^2-1) + \dots + (x^n-1)$

$= (x-1)[1 + (x+1) + (x^2+x+1) + \dots + (x^{n-1} + \dots + x + 1)]$

$$\lim_{x\to 1}\frac{x+x^2+\ldots+x^n-n}{x-1}=\lim_{x\to 1}\left[1+(x+1)+(x^2+x+1)+\ldots+(x^{n-1}+\ldots+x+1)\right]=\frac{n(n+1)}{2}$$

(d) If $m > n$,

$$\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\lim_{x\to1}\frac{m(1+x+x^2+\dots+x^{n-1})-n(1+x+x^2+\dots+x^{m-1})}{(1-x)(1+x+x^2+\dots+x^{m-1})(1+x+x^2+\dots+x^{n-1})}$$, on cancelling $(1 - x)$.

$$=\lim_{x\to1}\frac{m\left[(1+x+x^2+\dots+x^{n-1})-nx^{n-1}\right]-n\left[(1+x+\dots+x^{m-1})-mx^{m-1}\right]+mnx^{n-1}\left(1-x^{m-n}\right)}{(1-x)(1+x+x^2+\dots+x^{m-1})(1+x+x^2+\dots+x^{n-1})}$$

$$=\lim_{x\to1}\frac{m\left[1+2x+3x^2+\dots+(n-1)x^{n-2}\right]-nx^n\left[1+2x+3x^2+\dots+(m-1)x^{m-2}\right]+mnx^{n-1}(1+x+\dots+x^{m-n-1})}{(1+x+x^2+\dots+x^{m-1})(1+x+x^2+\dots+x^{n-1})}$$

, on cancelling $(1 - x)$.

$$=\frac{m[1+2+3+\dots+(n-1)]-n[1+2+3+\dots+(m-1)]+mn(m-n)}{mn}$$

$$=\frac{m(n-1)n/2-n(m-1)m/2+mn(m-n)}{mn}=\frac{(n-1)+(m-1)+2(m-n)}{2}=\frac{m-n}{2}$$

If $m < n$, $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=-\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{m}{1-x^m}\right)=-\frac{n-m}{2}=\frac{m-n}{2}$

If $m = n$, the limit is obviously equal to 0.

4. We like to prove first (i) $\lim_{x\to0^-} e^{\frac{1}{x}}=0$ (ii) $\lim_{x\to0^+} e^{\frac{1}{x}}=\infty$

(i) For any $\varepsilon > 0$, we like to find suitable $\delta > 0$ s.t. $e^{1/x} < \varepsilon$ ….(1)

Taking logarithm, we have $1/x < \ln \varepsilon$ ….(2)

Since $x \to 0^-$, we consider negative x values only .

For $\varepsilon \ge 1$, (2) always holds, we can choose any δ .

For $\varepsilon < 1$, $x < 0$, $\ln \varepsilon < 0$, multiply (2) by $x / \ln \varepsilon > 0$, we have $x > 1/ \ln \varepsilon$

or $x < 0 – x < - 1/ \ln \varepsilon$. $\therefore$ take $\delta = 1/ \ln \varepsilon$, (1) holds.

(ii) For any $R > 0$, we like to find suitable $\delta > 0$ s.t. $e^{1/x} > R$ ….(1)

For $R \le 1$, (1) always holds, we choose any δ .

For $R > 1$, taking logarithm in (1) , $0 < x-0 < \frac{1}{\log R}$. $\therefore$ Take $\delta=\frac{1}{\log R}$, (1) follows.

(a) $\lim_{x\to1^-}\frac{1}{1+e^{\frac{1}{x}}}=\frac{1}{1+0}=1$, by (i) **(b)** $\lim_{x\to1^+}\frac{1}{1+e^{\frac{1}{x}}}=\frac{1}{1+\infty}=0$, by (ii)

(c) Replace x by $1/x$. As $x \to -\infty$, $1/x \to 0^-$.

$$\lim_{x\to-\infty}\frac{\ln\left(1+e^x\right)}{x}=\lim_{x\to0^-}x\ln\left(1+e^{1/x}\right)=0\times\ln(1+0)=0$$

(d) $$\lim_{x\to+\infty}\frac{\ln\left(1+e^x\right)}{x}\underset{\text{L'hospital Rule}}{=}\lim_{x\to+\infty}\frac{\frac{e^x}{1+e^x}}{1}=\lim_{x\to0^+}\frac{e^{1/x}}{1+e^{1/x}}=\lim_{x\to0^+}\frac{1}{\frac{1}{e^{1/x}}+1}=\frac{1}{0+1}=1$$

5. **(a)** $$\lim_{x\to+\infty}\left[\sqrt{(x+a)(x+b)}-x\right]=\lim_{x\to+\infty}\frac{\left[\sqrt{(x+a)(x+b)}-x\right]\left[\sqrt{(x+a)(x+b)}+x\right]}{\sqrt{(x+a)(x+b)}+x}=\lim_{x\to+\infty}\frac{(x+a)(x+b)-x^2}{\sqrt{(x+a)(x+b)}+x}$$

$$=\lim_{x\to+\infty}\frac{x(a+b)+ab}{\sqrt{(x+a)(x+b)}+x}=\lim_{x\to+\infty}\frac{a+b+(ab/x)}{\sqrt{(1+(a/x))(1+(b/x)}+1}=\frac{a+b+0}{\sqrt{(1+0)(1+0}+1}=\frac{a+b}{2}$$

(b) $\lim_{x\to+\infty}\left(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1}\right)=\lim_{x\to+\infty}\frac{(x^3+x^2+1)-(x^3-x^2+1)}{(x^3+x^2+1)^{2/3}+(x^3+x^2+1)^{1/3}(x^3-x^2+1)^{1/3}+(x^3-x^2+1)^{2/3}}$

$$=\lim_{x\to+\infty}\frac{2x^2}{(x^3+x^2+1)^{2/3}+(x^3+x^2+1)^{1/3}(x^3-x^2+1)^{1/3}+(x^3-x^2+1)^{2/3}}$$

$$=\lim_{x\to+\infty}\frac{2}{\left(1+\frac{1}{x}+\frac{1}{x^3}\right)^{2/3}+\left(1+\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}\left(1-\frac{1}{x}+\frac{1}{x^3}\right)^{1/3}+\left(1-\frac{1}{x}+\frac{1}{x^3}\right)^{2/3}}=\frac{2}{1+1+1}=\frac{2}{3}$$

(c) $\lim_{x\to\infty}x^{\frac{1}{3}}\left[(x+1)^{2/3}-(x-1)^{2/3}\right]=\lim_{x\to\infty}x^{\frac{1}{3}}\frac{(x+1)^2-(x-1)^2}{(x+1)^{4/3}+(x+1)^{2/3}(x-1)^{2/3}+(x-1)^{4/3}}$

$$=\lim_{x\to\infty}\frac{4x^{4/3}}{(x+1)^{4/3}+(x+1)^{2/3}(x-1)^{2/3}+(x-1)^{4/3}}=\lim_{x\to\infty}\frac{4}{\left(1+\frac{1}{x}\right)^{4/3}+\left(1+\frac{1}{x}\right)^{2/3}\left(1-\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{4/3}}$$

$$=\frac{2}{1+1+1}=\frac{2}{3}$$

(d) $\lim_{x\to 64}\frac{\sqrt[3]{x}-4}{\sqrt{x}-8}\underset{y=x^6}{=}\lim_{y\to 2}\frac{y^2-4}{y^3-8}=\lim_{y\to 2}\frac{y+2}{y^2+2y+4}=\frac{4}{4+4+4}=\frac{1}{3}$

(e) $\lim_{x\to 8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}\underset{y=x^3}{=}\lim_{y\to 2}\frac{\sqrt{9+2y^3}-5}{y-2}=\lim_{y\to 2}\frac{\left(\sqrt{9+2y^3}-5\right)\left(\sqrt{9+2y^3}+5\right)}{(y-2)\left(\sqrt{9+2y^3}+5\right)}=\lim_{y\to 2}\frac{9+2y^3-25}{(y-2)\left(\sqrt{9+2y^3}+5\right)}$

$$=\lim_{y\to 2}\frac{2y^3-16}{(y-2)\left(\sqrt{9+2y^3}+5\right)}=\lim_{y\to 2}\frac{2\left(y^2+2y+4\right)}{\sqrt{9+2y^3}+5}=\frac{2(4+4+4)}{\sqrt{9+16}+5}=\frac{12}{5}$$

6. $\lim_{x\to a}f(x)=L \quad\Rightarrow\quad \forall\varepsilon>0,\exists\delta(\varepsilon)>0$ s.t. $0<|x-a|<\delta\Rightarrow|f(x)-L|<\varepsilon\Rightarrow\big||f(x)|-|L|\big|<|f(x)-L|<\varepsilon$

$\therefore\quad \lim_{x\to a}|f(x)|=|L|$

It is obvious that $\lim_{x\to a}f(x)=0\Rightarrow\lim_{x\to a}|f(x)|=0$ since $|f(x)|=\big||f(x)|\big|<\varepsilon$. If $L\neq 0$, counterexample :

Consider $f(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}$. $\therefore \lim_{x\to 0}|f(x)|=0,\quad \lim_{x\to 0}f(x)$ does not exist.

7. As x ranges through the sequence $\{x_{n_i}\}=\left\{n\pi+\frac{\pi}{2}\right\}$, n = 0, 1, 2, ….

$$\lim_{x\to\infty}f(x)=\lim_{n\to\infty}\frac{x_{n_i}\tan x_{n_i}+2x_{n_i}-1}{x_{n_i}+1}=\lim_{n\to\infty}\frac{\tan x_{n_i}+2-\frac{1}{x_{n_i}}}{1+\frac{1}{x_{n_i}}}=\frac{\infty+2-0}{1+0}=\infty$$

As x ranges through the sequence $\{x_{n_j}\}=\left\{n\pi+\frac{\pi}{4}\right\}$, n = 0, 1, 2, ….

$$\lim_{x\to\infty}f(x)=\lim_{n\to\infty}\frac{x_{n_j}\tan x_{n_j}+2x_{n_i}-1}{x_{n_j}+1}=\lim_{n\to\infty}\frac{\tan x_{n_j}+2-\frac{1}{x_{n_j}}}{1+\frac{1}{x_{n_j}}}=\frac{1+2-0}{1+0}=3$$

$\therefore$ f(x) does not have a unique limit as $x\to\infty$.

8. When x is an integer, $\sin^2 \pi x = 0$. $\therefore \lim_{n\to\infty} \frac{f(x) + ng(x)\sin^2 \pi x}{1 + n\sin^2 \pi x} = \lim_{n\to\infty} \frac{f(x)+0}{1+0} = f(x)$

When x is not an integer, $\sin^2 \pi x \neq 0$.

$$\therefore \lim_{n\to\infty} \frac{f(x) + ng(x)\sin^2 \pi x}{1 + n\sin^2 \pi x} = \lim_{n\to\infty} \frac{(1/n)f(x) + g(x)\sin^2 \pi x}{(1/n) + \sin^2 \pi x} = \frac{0 + g(x)\times 1}{0+1} = g(x).$$

9. When $x^2 > 1$, $|x| > 1$, $|1/x| < 1$, $\therefore (1/x)^n \to 0$, as $n \to \infty$.

$$\therefore \lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1} = \lim_{n\to\infty} \frac{f(x) + (1/x^n)g(x)}{1 + (1/x^n)} = \frac{f(x) + 0\times g(x)}{1+0} = f(x)$$

When $x^2 < 1$, $|x| < 1$, $\therefore$ $x^n \to 0$, as $n \to \infty$.

$$\therefore \lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1} = \frac{0\times f(x) + g(x)}{0+1} = g(x)$$

When $x = 1$. $\lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1} = \lim_{n\to\infty} \frac{1^n \times f(x) + g(x)}{1^n + 1} = \frac{f(x) + g(x)}{2}$.

$\therefore$ $\phi(x) = \lim_{n\to\infty} \frac{x^n f(x) + g(x)}{x^n + 1}$, $x \neq -1$. $\phi(x)$ is not continuous on **R** unless $f(x) = g(x)$.

10. $\lim_{x\to\infty} f(x) = 0 \Rightarrow$ $\forall \varepsilon > 0, \exists X > 0, \text{ s.t. } x > X \Rightarrow |f(x)| < \varepsilon$

Take $\varepsilon = \frac{1}{M}$, then $\forall M > 0, \exists X > 0, \text{ s.t. } x > X \Rightarrow \left|\frac{1}{f(x)}\right| > \frac{1}{\varepsilon} = M$, $f(x) \neq 0$ $\therefore \lim_{x\to\infty} \left|\frac{1}{f(x)}\right| = \infty$.

11. $\lim_{x\to\infty} f(x) = \infty \Rightarrow$ $\forall M > 0, \exists X > 0, \text{ s.t. } x > X \Rightarrow |f(x)| > \sqrt{M}$ $\Rightarrow |f(x)|^2 > M$

$\therefore$ $\lim_{x\to\infty} [f(x)]^2 = \infty$.

12. No. Consider $f(x) = 1 - \frac{1}{x^2}$. $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = 1$.

13. Yes. $\lim_{x\to a} f(x) = \infty \Rightarrow$ $\forall M > 0, \exists \delta > 0, \text{ s.t. } |x - a| < \delta \Rightarrow |f(x)| > M$

$\Rightarrow f(x) < -M \text{ or } f(x) > M$ $\Rightarrow f(x) < -M$ $\therefore \lim_{x\to a} f(x) = -\infty$

But $\lim_{x\to a} f(x) = +\infty$ does not imply $\lim_{x\to a} f(x) = -\infty$. Counterexample : $f(x) = \frac{1}{(x-a)^2}$.

14. No, unless $m = n$.

$\lim_{x\to a} f(x) = m \Rightarrow$ $\forall \delta_1 > 0,\ 0 < |x - a| < \delta_1 \Rightarrow |f(x) - m| < \varepsilon/2$

$\lim_{x\to a} f(x) = n \Rightarrow$ $\forall \delta_2 > 0,\ 0 < |x - a| < \delta_2 \Rightarrow |f(x) - n| < \varepsilon/2$

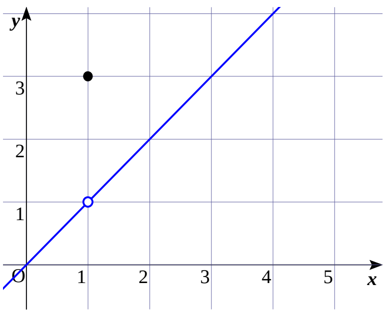
Take $\delta = \min(\delta_1, \delta_2)$ $0 < |x - a| < \delta \Rightarrow |m - n| = |(f(x) - m) - (n - f(x)| < |f(x) - m| + |f(x) - n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since ε is arbitrary, $|m - n| = 0$ $\Rightarrow m = n$.

15. No. unless $f(x)$ is continuous in $x = a$.

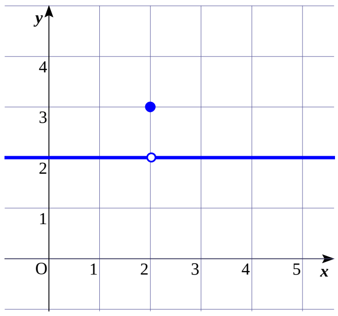
Counterexample : $f(x) = \begin{cases} 1 & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$.

16.



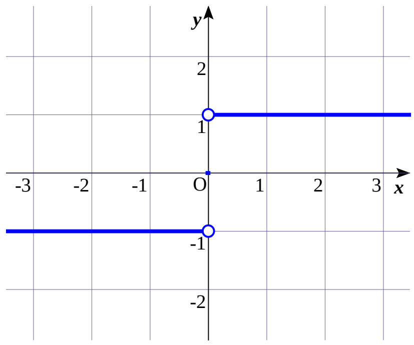
$$\lim_{x\to 1} f(x) = 1 \neq f(1) = 3$$

17.



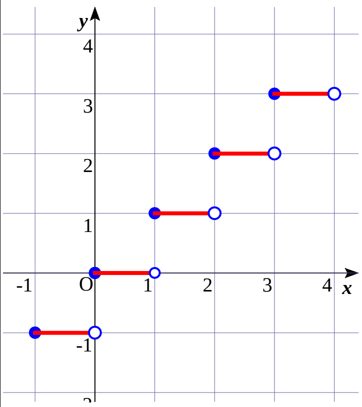
$$\lim_{x\to 2} f(x) = 2 \neq f(2) = 3$$

18.



$$\lim_{x\to a} \operatorname{sgn}(x) = \begin{cases} 1 & \text{if} \quad a > 0 \\ -1 & \text{if} \quad a < 0 \\ \text{undefined} & \text{if} \quad a = 0 \end{cases}$$

19.



$$\lim_{x\to a}[x] = \begin{cases} [x] & \text{if} \quad x \notin \mathbf{Z} \\ \text{undefined} & \text{if} \quad x \in \mathbf{Z} \end{cases}$$

20. **(a)** Let $P(n) : n \geq 10, \quad 2^n > n^3$.

For $P(10)$, $\quad 2^{10} = 1024 > 1000 = 10^3$. $\quad \therefore \quad P(10)$ is true.

Assume $P(k)$ is true for some $k \in \mathbf{N}$, $k \geq 10$.

i.e. $\quad 2^k > k^3 \qquad \ldots.(*)$

For $P(k+1)$, $\quad 2^{k+1} = 2(2^k) = (k+1)^3 + k^3 - 3k^2 - 3k - 1$, by (*)

$> (k+1)^3 + 10k^2 - 3k^2 - 3k - 1 \quad$ (since $k \geq 10$)

$= (k+1)^3 + 7k^2 - 3k - 1 > (k+1)^3 + 70k - 3k - 1$

$= (k+1)^3 + 67k - 1 > (k+1)^3 + 670 - 1 > (k+1)^3$

$\therefore$ $P(k+1)$ is true. Result follows from Principle of Math. Induction.

(b) Let $n = [x]$. Then $n + 1 > x \geq n$. $\quad \dfrac{2^x}{3x^2} \geq \dfrac{2^n}{3(n+1)^2} > \dfrac{n^3}{3(n+1)^2}$

$$\therefore \lim_{n\to\infty} \frac{2^x}{3x^2} \geq \lim_{n\to\infty} \frac{n^3}{3(n+1)^2} = \lim_{n\to\infty} \frac{n}{3\left(1+\frac{1}{n}\right)^2} = \infty \qquad \therefore \lim_{n\to\infty} \frac{2^x}{3x^2} = \infty$$

21. **(a)** $$\lim_{x\to 0} \frac{x^2 - 1}{2x^2 - x - 1} = \frac{0 - 1}{0 - 0 - 1} = 1$$

(b) $$L = \lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x) - 1}{x} = \lim_{x\to 0} \frac{6x + 11x^2 + 6x^3}{x} = \lim_{x\to 0}\left(6 + 11x + 6x^2\right) = 6$$

Also, $$L \underset{LHR}{=} \lim_{x\to 0}[(1+2x)(1+3x) + 2(1+3x)(1+x) + 3(1+x)(1+2x)] = 6$$

(c) $$\lim_{t\to 1} \frac{t^2 - t^{\frac{1}{2}}}{t^{\frac{1}{2}} - 1} \underset{x^2 = t}{=} \lim_{x\to 1} \frac{x^4 - x}{x - 1} = \lim_{x\to 1} x\left(x^2 + x + 1\right) = 3$$

(d) $$L = \lim_{x\to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2}$$

$$L = \lim_{x\to 0} \frac{\left[1 + n(mx) + \dfrac{n(n-1)}{2}(mx)^2 + \sum_{r=3}^{n} C_r^n (mx)^r\right] - \left[1 + m(nx) + \dfrac{m(m-1)}{2}(nx)^2 + \sum_{r=3}^{m} C_r^m (nx)^r\right]}{x^2}$$

$$= \lim_{x\to 0}\left\{\left[\frac{n(n-1)}{2}(m)^2 + \sum_{r=3}^{n} C_r^n m^r x^{r-2}\right] - \left[\frac{m(m-1)}{2}(n)^2 + \sum_{r=3}^{m} C_r^m n^r x^{r-2}\right]\right\}$$

$$= \frac{n(n-1)}{2}(m)^2 - \frac{m(m-1)}{2}(n)^2 = \frac{mn(m-n)}{2}.$$ Also, by L'hospital Rule,

$$L = \lim_{x\to 0}\frac{mn(1+mx)^{n-1} - mn(1+nx)^{m-1}}{2x} = \lim_{x\to 0}\frac{m^2n(1+mx)^{n-2} - mn^2(1+nx)^{m-1}}{2} = \frac{mn(m-n)}{2}$$

(e) $$\lim_{x\to 0}\frac{\sin mx}{\sin nx} = \frac{m}{n}\lim_{x\to 0}\frac{\sin mx}{mx}\Big/\lim_{x\to 0}\frac{\sin nx}{nx} = \frac{m}{n}$$

(f) $$\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} = -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h/2} = -\sin x$$

(g) $$\lim_{x\to -\infty}\left(\sqrt{x^2+1}-x\right) \underset{y=-x}{=} \lim_{y\to +\infty}\left(\sqrt{(-y)^2+1}-(-y)\right) = \lim_{y\to +\infty}\left(\sqrt{y^2+1}+y\right) = +\infty$$

(h) $$\lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}} = \lim_{x\to\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x^2}+\sqrt{\frac{1}{x^4}}}}}{\sqrt{1+\frac{1}{x}}} = 1$$

(i) $$\lim_{x\to 0}\frac{1+\sin x-\cos x}{1+\sin px-\cos px} \underset{LHR}{=} \lim_{x\to 0}\frac{\cos x-\sin x}{p\cos px - p\sin px} = \frac{1-0}{p-0} = \frac{1}{p}$$

(j) Let $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{i=0}^{n} b_i x^i$

$\lim_{x\to+\infty}\frac{p(x)}{q(x)} = \begin{cases} a_m/b_n & \text{if } m=n \\ 0 & \text{if } m<n \\ +\infty & \text{if } m>n \end{cases}$	$\lim_{x\to+\infty}\frac{p(x)}{q(x)} = \begin{cases} a_m/b_n & \text{if } m=n \\ 0 & \text{if } m<n \\ +\infty & \text{if } m>n, m \text{ is even} \\ -\infty & \text{if } m>n, m \text{ is odd} \end{cases}$

22. $f(x) = \begin{cases} 1 & \text{if } x<a \\ 0 & \text{if } x=a \\ 1 & \text{if } x>a \end{cases}$. $\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 1 \neq f(a) = 0$.

23. $f(x) = \begin{cases} \ln\frac{e}{(x-a)^2} & \text{if } x<a \\ 0 & \text{if } x=a \\ \ln\frac{e^2}{(x-a)^2} & \text{if } x>a \end{cases}$ $g(x) = \begin{cases} \ln\frac{1}{(x-a)^2} & \text{if } x<a \\ 0 & \text{if } x=a \\ \ln\frac{1}{(x-a)^2} & \text{if } x>a \end{cases}$. Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

But $f(x)-g(x) = \begin{cases} \ln e = 1 & \text{if } x<a \\ 0 & \text{if } x=a \\ \ln e^2 = 2 & \text{if } x>a \end{cases}$. $\therefore \lim_{x\to a}[f(x)-g(x)]$ does not exist.

24. If $f(x)\neq 0$ $\forall$ x, $0<|x-a|<\delta$, then $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty \Rightarrow \lim_{x\to a}\frac{g(x)}{f(x)} = \infty$.

Proof : $\lim_{x\to a} f(x) = 0$ $\therefore$ $\forall\varepsilon>0, \exists\delta_1>0,\ 0<|x-a|<\delta_1 \Rightarrow |f(x)|<\varepsilon$

Take $\varepsilon = \frac{1}{\sqrt{R}}$, $\forall R>0, \exists\delta_1>0,\ 0<|x-a|<\delta_1 \Rightarrow \left|\frac{1}{f(x)}\right| < \sqrt{R}$

Now, $\lim_{x\to a} g(x)=\infty \quad \therefore \forall R>0, \exists \delta_2>0, \quad 0<|x-a|<\delta_2 \Rightarrow |g(x)|<\sqrt{R}$

Take $\delta = \min(\delta_1, \delta_2)$, $\forall R>0, \exists \delta>0, \quad 0<|x-a|<\delta_2 \quad \Rightarrow \left|\frac{g(x)}{f(x)}\right| < \sqrt{R}\sqrt{R} = R$. $\quad \therefore \lim_{x\to a}\frac{g(x)}{f(x)}=\infty$

If $f(x)\equiv 0 \quad \forall x, \quad 0<|x-a|<\delta$, then $\frac{g(x)}{f(x)}$ is undefined and $\lim_{x\to a}\frac{g(x)}{f(x)}$ does not exist.

For example, $f(x)\equiv 0$, $g(x)=\frac{1}{x-a}$.

25. $f(x)=\begin{cases}1 & \text{if } x<a\\ -1 & \text{if } x\ge a\end{cases} \qquad |f(x)|=1, \qquad \therefore \lim_{x\to a}|f(x)|=1$, but $\lim_{x\to a} f(x)$ does not exist .

26. **(a)** $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^2+2x)=3, \qquad \lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 0 = 0$

(b) Both left and right limits do not exist.

27. **(a)** Let $x_n=\left(1+\frac{1}{n}\right)^n, \quad y_n=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}$, then

$x_n=1+\binom{n}{1}\frac{1}{n}+\ldots+\binom{n}{k}\frac{1}{n^k}+\ldots+\binom{n}{n}\frac{1}{n^n}$, where $k\le n$.

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{n-1}{n}\right) \ldots(1)$$

Since $k<n$, $x_n>2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)$

Let $n\to\infty$, $e\ge 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}=y_n$.

Clearly, $e\ge y_n>x_n$, since in (1), $\left(1-\frac{1}{n}\right),\left(1-\frac{2}{n}\right),\ldots,\left(1-\frac{k-1}{n}\right),\ldots,\left(1-\frac{n-1}{n}\right)$ are all less than 1.

and so by Sandwich theorem, $\lim_{n\to\infty} y_n=\lim_{n\to\infty}\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}\right)=e$ exists.

(b) In (1), $x_n=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)$

$$+\frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\left[\frac{1}{k+1}\left(1-\frac{k}{n}\right)+\ldots+\frac{1}{(k+1)(k+2)\ldots n}\left(1-\frac{k}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)\right]$$

$$<1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!}+\frac{1}{k!}\left[\frac{1}{k+1}+\frac{1}{(k+1)^2}+\ldots+\frac{1}{(k+1)^{n-k}}\right]<y_k+\frac{1}{k!}\left[\frac{1}{k+1}+\frac{1}{(k+1)^2}+\ldots\right]$$

$$<y_k+\frac{1}{k!}\frac{1}{k+1}\frac{1}{1-\frac{1}{k+1}}=y_k+\frac{1}{k\times k!}$$

Let $n\to\infty \quad \therefore\ e\le y_k+\frac{1}{k\times k!} \qquad \therefore e=y_k+\frac{\theta_k}{k\times k!}, 0<\theta_k<1$.

Replace k by n, $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\frac{\theta_n}{n!n}$, where $0<\theta_n<1$.

(c) Take $n = 10$ in (b), $e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{10!} \approx 2.718281 \approx 2.71828$ (to 5 dec. places)

$$\frac{\theta_n}{n!n} = \frac{\theta_{10}}{10! \times 10} < \frac{1}{10! \times 10} < 0.000,000,1$$

(d) **(i)** $$\lim_{n\to\infty}\left(1-\frac{2}{3n}\right)^n = \left[\lim_{n\to\infty}\left(1+\frac{1}{(-3n/2)}\right)^{(-3n/2)}\right]^{-2/3} = e^{-2/3} = \frac{1}{e^{2/3}}$$

(assuming the limit is good for continuous and negative variable)

(ii) Put $y = \frac{x^2 - a}{2a}$. Then $x \to \infty$, $y \to \infty$.

$$\lim_{x\to\infty}\left(\frac{x^2+a}{x^2-a}\right)^{x^2} = \lim_{x\to\infty}\left(\frac{2ay+2a}{2ay}\right)^{2ay+a} = \lim_{x\to\infty}\left(1+\frac{1}{y}\right)^{2ay+a} = \left[\lim_{x\to\infty}\left(1+\frac{1}{y}\right)^y\right]^{2a}\lim_{x\to\infty}\left(1+\frac{1}{y}\right)^a = e^a$$

(iii) $$\lim_{n\to\infty}\left(\frac{n}{a+n}\right)^n = \left[\lim_{n\to\infty}\left(1+\frac{1}{-(a+n)/a}\right)^{-(a+n)/a}\right]^{-a}\lim_{n\to\infty}\left(1+\frac{1}{-(a+n)/a}\right)^{-a} = e^{-a}\times 1 = e^{-a}$$

(iv) The question is easy if the limit is good for negative variable. If the variable is restricted to positive:

$$\lim_{x\to 0}(1-x)^{1/x} = \lim_{y\to\infty}\left(1-\frac{1}{y}\right)^y = \lim_{y\to\infty}\left(\frac{y-1}{y}\right)^y = \lim_{y\to\infty}\left(\frac{y}{y-1}\right)^{-y} = \left[\lim_{y\to\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\right]^{-1} \Big/ \lim_{y\to\infty}\left(1+\frac{1}{y-1}\right) = \underline{\underline{e^{-1}}}$$

(e)

$$\lim_{n\to\infty}\left(1+\frac{1}{P_n}\right)^{P_n} = \lim_{P_n\to\infty}\left(1+\frac{1}{P_n}\right)^{P_n} = \underline{\underline{e}}$$

(f)

$$\lim_{n\to\infty}\left(\frac{an+b}{an+c}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{an+c}{b-c}}\right)^{\frac{an+c}{b-c}\times\frac{b-c}{a}-\frac{c}{b}} = \underline{\underline{e^{\frac{b-c}{a}}}}$$

(g)

$$\lim_{x\to 0}(1+x)^{\frac{b}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{by} = \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y\right]^b = e^b$$

Replace x by $1/x$,

$$\lim_{x\to\infty}\left(\frac{1-\frac{2}{x}-\frac{3}{x^2}}{1-\frac{3}{x}-\frac{28}{x^2}}\right)^x = \lim_{x\to 0}\left(\frac{1-2x-3x^2}{1-3x-28x^2}\right)^{1/x} = \lim_{x\to 0}\left(\frac{(1-3x)(1+x)}{(1-7x)(1+4x)}\right)^{1/x}$$

$$\lim_{x\to 0}(1+(-3x))^{(-3)/(-3x)}\lim_{x\to 0}(1+x)^{1/x}\Big/\lim_{x\to 0}(1+(-7x))^{(-7)/(-7x)}\lim_{x\to 0}(1+4x)^{4/(4x)} = e^{-3}e/e^{-7}e^4 = \underline{\underline{e}}$$